ODEs: Symplectic integrators and Orbits

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1 Orbits

As a starting place, lets derive the ODEs for two objects orbiting each other. We will assume these objects only move in the x-y plane for simplicity. The gravitational force between our two objects is given by

$$\vec{F} = \frac{GM_1M_2}{r^2}\hat{r}_{12} \tag{1}$$

Here M_1 and M_2 are the masses of our objects, r is the distance between the objects, and $\hat{r_{12}}$ is a unit vector (length of one) that points from mass one towards mass two. For any objects in the x-y plane we can solve for r via

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{2}$$

where (x_1, y_1) would be the location of the first mass. $\hat{r}_{12}$ is giving us information about how much of this force is in the x direction versus the y direction. Equivalently you can think of this as how much of the is the x or y direction. Because of this we can write

$$\hat{r}_{12} = \left(\frac{x_2 - x_1}{r}\right)\hat{x} + \left(\frac{y_2 - y_1}{r}\right)\hat{y} \tag{3}$$

Because we have components of our force in the x and y directions we will use separate equations for each direction. With this we can write our system of equations as

$$\frac{dx_1}{dt} = v_{x,1} \tag{4}$$

$$\frac{dy_1}{dt} = v_{y,1} \tag{5}$$

$$\frac{dx_2}{dt} = v_{x,2} \tag{6}$$

$$\frac{dy_2}{dt} = v_{y,2} \tag{7}$$

$$\frac{dv_{x,1}}{dt} = \frac{GM_2}{r^2}\left(\frac{x_2 - x_1}{r}\right) \tag{8}$$

$$\frac{dv_{y,1}}{dt} = \frac{GM_2}{r^2}\left(\frac{y_2 - y_1}{r}\right) \tag{9}$$

$$\frac{dv_{x,2}}{dt} = \frac{GM_1}{r^2}\left(\frac{x_1 - x_2}{r}\right) \tag{10}$$

$$\frac{dv_{y,2}}{dt} = \frac{GM_1}{r^2}\left(\frac{y_1 - y_2}{r}\right) \tag{11}$$

So for two objects orbiting each other we have 8 equations. This means we will need 8 initial conditions to start off our system. Note that if we included another object we would need to add an additional 4 equations of motion for that object as well as including the force between that objects and the other two.

The advantage of doing the two body system first is we have the true solution to compare to. If one of the objects is significantly more massive than the other (say the Sun and Earth) we can describe the orbit as an ellipse. In this limit the semi-major axis, a, is the mean distance between the two objects and the eccentricity describes the deviation of the ellipse from a circle. The eccentricity can vary between zero and one where $e = 0$ is a circle and $e = 1$ is a parabola. The position of the smaller object in this case can be described by

$$r(\theta) = \frac{a(1 - e^2)}{1 - e\cos\theta} \tag{12}$$

Here θ is the true anomaly of the orbit. True anomaly is a angular measure (radians) that describes how far the object is from its apoapsis (the farthest point from the sun). We can also analyticly get the orbital period is

$$\tau = \sqrt{\frac{4\pi^2 a^3}{G(M_1 + M_2)}} \tag{13}$$

1.1 Initial Conditions

While we could solve this system for any initial conditions using some basic physics we can pick a start that will keep our objects in orbit. Let us set the center of our coordinate system as the system center of mass. We will rotate our coordinate such that both objects are on the x axis (i.e. their initial y-coordinate is 0). The starting x position of each object is

$$x_1(0) = r_0 \tag{14}$$
$$x_2(0) = -qr_0 \tag{15}$$

Here $q = M_1/M_2$. r_0 is given by

$$r_0 = \frac{1-e}{1+q}a \tag{16}$$

Using a little algebra we can also calculate the initial velocity in this frame is

$$v_{y,1} = v0 \tag{17}$$
$$v_{y,2} = -qv_0 \tag{18}$$
$$v_0 = \frac{1}{1+q}\sqrt{\frac{1+e}{1-e}}\sqrt{\frac{G(M_1+M_2)}{a}} \tag{19}$$

Try to solve these equations with your midpoint or RK solver. What happens after 100 or 1000 orbits? Also try changing dt to see how that changes the result.

2 Leapfrog integrator

A leap from integrator works by updating our knowledge of the velocity and position a half step apart from one another. So we update our knowledge of x via

$$x_{n+1} = x_n + \Delta t v_{n+1/2} \tag{20}$$

Then we will update our knowledge of v via

$$v_{n+1/2} = v_{n-1/2} + \Delta t a(x_n) \tag{21}$$

where $a(x)$ is the acceleration and is only a function of position. This is important because it is actually equivalent to stating energy is conserved (this is nicely explained by this minutephysics video).

Historical aside: The fact that the having equations that are symmetric in time is equivalent to saying energy is conserved was discovered by Emmy Noether in 1915. She did this while doing research and teaching but received no pay for her 3 years of work at University of Göttingen because she was a woman. This worked shaped much of modern physics and is the reason modern physicists are so interested in equations that have symmetry.

By taking these "leapfrog" steps, this integrator is symmetric in time meaning you could run it backwards and get where you started. This is not generally true of Eulerian or Runge-Kutta solvers. Because of this symmetry it conserves an "energy like quantity". To give an explanation much deeper into this topic requires Hamiltonian physics.