ODEs:RK solver and Pendulums

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1 Midpoint

Last class we used an Eulerian solver which used the formula

$$x_{n+1} = x_n + \Delta t g(x_n, t_n) \tag{1}$$

to solve the equation

$$\frac{dx}{dt} = g(x, t). \tag{2}$$

A midpoint solver uses the same general idea as the Eulerian solver but instead of taking one big jump each time step, we take a trial step at the midpoint before making the jump. This can be written as

$$k_1 = \Delta t g(x_n, t_n) \tag{3}$$

$$k_2 = \Delta t g(x_n + \frac{1}{2}k_1, t_n + \frac{1}{2}\Delta t) \tag{4}$$

$$x_{n+1} = x_n + k_2 \tag{5}$$

In a sense this can be thought of as including 1 higher order term in a Taylor series expansion of the original differential equation. This is why Euler's method is accurate to first order, $\mathcal{O}(\Delta t^2)$ while the midpoint method is accurate to within $\mathcal{O}(\Delta t^3)$.

2 Runge-Kutta

There is no reason to stop at one point in the middle. One very common integrator is the fourth-order Runge-Kutta (or RK4). The set of equations we need for RK4 is

$$k_1 = \Delta t g(x_n, t_n) \tag{6}$$

$$k_2 = \Delta t g(x_n + \frac{1}{2}k_1, t_n + \frac{1}{2}\Delta t) \tag{7}$$

$$k_3 = \Delta t g(x_n + \frac{1}{2}k_2, t_n + \frac{1}{2}\Delta t) \tag{8}$$

$$k_4 = \Delta t g(x_n + k_3, t_n + \Delta t) \tag{9}$$

$$x_{n+1} = x_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \tag{10}$$

RK4 is generally considered the solver that will give you the best precision without doing too many calculation. As will will see going forward however it is not perfect and will fail in some situations.

3 Single Pendulum

A pendulum has its x and y position defined by

$$x = l\sin\theta \tag{11}$$
$$y = -l\cos\theta \tag{12}$$

where l is the length of the pendulum and θ is the angle away from vertical (the rest state). This tells us that we can fully describe the position of the pendulum by one variable, θ. The restoring force on the pendulum is given by

$$F_{net} = -mg\sin\theta\hat{\theta} \tag{13}$$

where $\hat{\theta}$ is the unit vector in the θ direction.

From this we can get equations of motion that are

$$\dot{\theta} = \omega \tag{14}$$
$$\ddot{\theta} = \dot{\omega} = -\frac{g}{l}\sin\theta \tag{15}$$

4 Double Pendulum

For the double pendulum we have another pendulum on the end of our first. This system is defined by 2 angles that define the system

$$x_1 = l_1\sin\theta_1 \tag{16}$$
$$y_1 = -l_1\cos\theta_1 \tag{17}$$
$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \tag{18}$$
$$y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2 \tag{19}$$

You can solve for the equations of motion for this system via conservation of energy. I will skip the details and just state the equations are

$$\dot{\theta}_1 = \frac{l_2 p_1 - l_1 p_2\cos(\theta_1-\theta_2)}{l_1^2 l_2\left(m_1 + m_2\sin^2(\theta_1-\theta_2)\right)} \tag{20}$$
$$\dot{\theta}_2 = \frac{l_1\left(m_1+m_2\right)p_2 - l_2 m_2 p_1\cos(\theta_1-\theta_2)}{l_1 l_2^2 m_2\left(m_1 + m_2\sin^2(\theta_1-\theta_2)\right)} \tag{21}$$
$$\dot{p}_1 = -(m_1+m_2)gl_1\sin(\theta_1) - C_1 + C_2 \tag{22}$$
$$\dot{p}_2 = -m_2 g l_2\sin\theta_2 + C_1 - C_2 \tag{23}$$
$$C_1 = \frac{p_1 p_2\sin(\theta_1-\theta_2)}{l_1 l_2\left[m_1 + m_2\sin^2(\theta_1-\theta_2)\right]} \tag{24}$$
$$C_2 = \frac{l_2^2 m_2 p_1^2 + l_1^2(m_1+m_2)p_2^2 - l_1 l_2 m_2 p_1 p_2\cos(\theta_1-\theta_2)}{2l_1^2 l_2^2\left[m_1 + m_2\sin^2(\theta_1-\theta_2)\right]^2}\sin\left(2\left(\theta_1-\theta_2\right)\right) \tag{25}$$

here p_1 and p_2 are the momenta of the pendulums (momentum is not conserved in this system).